

Flow equation for Halpern-Huang directions of scalar $O(N)$ models

Holger Gies

*Institut für theoretische Physik, Universität Tübingen,
72076 Tübingen, Germany*

February 1, 2008

Abstract

A class of asymptotically free scalar theories with $O(N)$ symmetry, defined via the eigenpotentials of the Gaussian fixed point (Halpern-Huang directions), are investigated using renormalization group flow equations. Explicit solutions for the form of the potential in the nonperturbative infrared domain are found in the large- N limit. In this limit, potentials without symmetry breaking essentially preserve their shape and undergo a mass renormalization which is governed only by the renormalization group distance parameter; as a consequence, these scalar theories do not have a problem of naturalness. Symmetry-breaking potentials are found to be “fine-tuned” in the large- N limit in the sense that the nontrivial minimum vanishes exactly in the limit of vanishing infrared cutoff: therefore, the $O(N)$ symmetry is restored in the quantum theory and the potential becomes flat near the origin.

1 Introduction

Common belief holds that only polynomial interactions up to a certain degree depending on the spacetime dimension are renormalizable, in the sense that interactions of even higher order require an infinite number of subtractions in a perturbative analysis. This can be attributed to the implicit assumption that the higher-order couplings, which in general are dimensionful, set independent scales. Such nonrenormalizable theories can only be defined with a cutoff scale Λ , while the unknown physics beyond the cutoff is encoded in the (thereby independent) values of the couplings.

Starting from the viewpoint that the cutoff Λ is the only scale in the theory, Halpern and Huang [1, 2] pointed out the existence of theories with higher-order and even nonpolynomial interactions within the conventional setting of quantum field theory. This happens because the higher-order couplings, by assumption, are proportional to a corresponding power of $1/\Lambda$ and therefore die out sufficiently fast in the limit $\Lambda \rightarrow \infty$; the theories remain perturbatively renormalizable in the sense that infinitely many subtractions are

not required. Perhaps most important, Halpern and Huang so discovered nonpolynomial scalar theories which are asymptotically free, offering an escape route to the “problem of triviality” of standard scalar theories [3].

To be more precise, Halpern and Huang analyzed the renormalization group (RG) trajectories for the interaction potential in the vicinity of the Gaussian fixed point. The exact form of the potential was left open by using a Taylor series expansion in the field as an ansatz. Employing the Wegner-Houghton [4] (sharp-cutoff) formulation of the Wilsonian RG, the eigenpotentials, i.e., tangential directions to the RG trajectories at the Gaussian fixed point, were identified in linear approximation. While the standard polynomial interactions turn out to be irrelevant as expected, some nonpolynomial potentials which increase exponentially for strong fields prove to be relevant perturbations at the fixed point. For the irrelevant interactions, the Gaussian fixed point is infrared (IR) stable, whereas the relevant ones approach this fixed point in the ultraviolet (UV). Possible applications of these new relevant directions are discussed in [1] for the Higgs model and in [5] for quintessence. Further nonpolynomial potentials and their applications in Higgs and inflationary models have been investigated in [6].

Considering the complete RG flow of such asymptotically free theories from the UV cutoff Λ down to the infrared, the Halpern-Huang result teaches us only something about the very beginning of the flow close to the cutoff and thereby close to the Gaussian fixed point. Each RG step in a coarse-graining sense “tends to take us out of the linear region into unknown territory” [2]. It is the purpose of the present work to perform a first reconnaissance of this territory with the aid of the RG flow equations for the “effective average action” [7]. In this framework, the standard effective action Γ is considered as the zero-IR-cutoff limit of the effective average action $\Gamma_k[\phi]$ which is a type of coarse-grained free energy with a variable infrared cutoff at the mass scale k . Γ_k satisfies an exact renormalization group equation, and interpolates between the classical action $S = \Gamma_{k \rightarrow \Lambda}$ and the standard effective action $\Gamma = \Gamma_{k \rightarrow 0}$.

In this work, we identify the classical action S given at the cutoff Λ with a scalar $O(N)$ symmetric theory defined by a standard kinetic term and a generally nonpolynomial potential of Halpern-Huang type. Therefore, we have the following scenario in mind: at very high energy, the system is at the UV stable Gaussian fixed point. As the energy decreases, the system undergoes an (unspecified) perturbation which carries it away from the fixed point initially into some tangential direction to one of all possible RG trajectories. We assume that this perturbation occurs at some scale Λ which then sets the only dimensionful scale of the system. Any other (dimensionless) parameter of the system should also be determined at Λ ; for the Halpern-Huang potentials, there are two additional parameters: one labels the different RG trajectories; the other specifies the “distance” scale along the trajectory. Finally, the precise form of the potential at Λ serves as the boundary condition for the RG flow equation which governs the behavior of the theory at all scales $k \leq \Lambda$.

Since the RG flow equations for Γ_k are equivalent to an infinite number of coupled differential equations of first order, a number of approximations (truncations) are necessary to arrive at explicit solutions. In the present work, we shall determine the RG trajectory $k \rightarrow \Gamma_k$ for $k \in [0, \Lambda]$ explicitly only in the large- N limit which simplifies the calculations

considerably.

The paper is organized as follows: Sec. 2, besides introducing the notation, briefly re-derives the Halpern-Huang result in the language of the effective average action, generalizing it to a nonvanishing anomalous dimension. Sec. 3 investigates the RG flow equation for the Halpern-Huang potentials in the large- N limit, concentrating on $d = 3$ and $d = 4$ space-time dimensions; here, we emphasize the differences to ordinary ϕ^4 theory particularly in regard to mass renormalization and symmetry-breaking properties. Sec. 4 summarizes our conclusions and discusses open questions related to finite values of N .

As an important caveat, it should be mentioned that the results of Halpern and Huang have been questioned (see [8] and also [9]), and these questions raised also affect the present work. To be honest, we have hidden the problems in the “scenario” described above in which an “unspecified” perturbation controls the shift of the system from the Gaussian fixed point (the continuum limit) to the cutoff scale Λ along a *tangential* direction. But since the cutoff scale Λ , though large, is not at all infinitesimally separated from the Gaussian fixed point, this tangential approximation is probably not sufficient to stay on the true renormalized trajectory during the shift. Not only the tangent but also all (infinitely many) curvature moments of the trajectory had to be known in order to find an initial point right on the renormalized trajectory at Λ . This point would correspond to a so-called “perfect action” [11]. Of course, this requires an infinite number of conditions to be imposed on the initial action at Λ which we cannot specify. In conventional field theories, this problem is solved by adjusting (fine-tuning) the initial action close to the unstable Gaussian fixed point, leaving open only one a priori chosen relevant direction to the flow. But in the present case, there is an infinite number of relevant directions corresponding to the continuum of possible Halpern-Huang directions, and thus it seems impossible to single out only one relevant direction while frustrating the others by tuning infinitely many parameters. In other words, upon studying the flow from Λ down to zero within our scenario, the continuum limit of our system remains unspecified, and therefore one important ingredient to a complete field theoretic system is missing.

With these reservations in mind, we nevertheless believe that there are some lessons to be learned from the application of the RG flow equations to such potentials.

2 Scalar $O(N)$ theories close to the Gaussian fixed point

Concerning the investigation of the RG flow equation for the Euclidean effective average action in d dimensions, we closely follow the original work of Wetterich [7]. Polynomial potentials and the large- N limit to be discussed later have been explored in [10] and [12] in the effective average action approach. A comprehensive review and an extensive list of references on this subject can be found in [13]. The effective average action can be

expanded in terms of all possible $O(N)$ invariants,

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \phi^b \partial_\mu \phi^b + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial_\mu \rho + \dots \right\}, \quad (1)$$

where $\rho := \frac{1}{2} \phi^b \phi^b$, $b = 1 \dots N$ labels the real components of the scalar field, and the dots represent terms involving higher derivatives; for convenience, we shall always assume that $d > 2$ during the calculation. Halpern and Huang derived their result in the “local-potential approximation” which is constituted by setting the wave function renormalization constant $Z_k \equiv 1$ and neglecting Y_k and higher-derivative terms. In the present work, we shall generalize their result to a k -dependent Z_k which is parametrized by the anomalous dimension,

$$\eta := -\partial_t \ln Z_k, \quad \text{where} \quad \partial_t \equiv k \frac{d}{dk} \quad (2)$$

denotes the derivative with respect to the RG “time”, $t \in]-\infty, 0]^1$ $t = \ln k/\Lambda$. Here we neglect Y_k and any ρ dependence of Z_k . Following [7], the RG flow equation for the effective average potential $U_k(\rho)$ can be written as

$$\partial_t U_k(\rho) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_t R_k \left(\frac{N-1}{Z_k q^2 + R_k + U'_k(\rho)} + \frac{1}{Z_k q^2 + R_k + U'_k + 2\rho U''_k(\rho)} \right), \quad (3)$$

where the prime denotes the derivative with respect to the argument ρ . The cutoff function $R_k = R_k(q^2)$ is to some extent an arbitrary positive function that interpolates between $R_k(q^2) \rightarrow Z_k k^2$ for $q^2 \rightarrow 0$ and $R_k(q^2) \rightarrow 0$ for $q^2 \rightarrow \infty$. It suppresses the small-momentum modes by a mass term k^2 acting as the IR cutoff. In Eq. (3) the distinction between the $N-1$ “Goldstone modes” and the “radial mode” is visible.

Provided that η is given (which we shall always assume in this work), the flow of the effective potential U_k (and thus of the effective action Γ_k in the present approximation) is determined by Eq. (3). Even if η is neglected, Eq. (3) produces qualitatively good results for polynomial effective potentials in $d > 2$ [13]. We expect similar behavior for nonpolynomial potentials.

The Halpern-Huang result can be rederived by assuming that the system is close to the Gaussian fixed point so that the effective potential and its derivatives are small. Linearizing the right-hand side of Eq. (3) with respect to the potential and its derivatives gives

$$\partial_t U_k(\rho) = -v_d (N U'_k(\rho) + 2\rho U''_k(\rho)) \int_0^\infty dw w^{d/2-1} \frac{\partial_t R_k(w)}{(Z_k w + R_k(w))^2} + \mathcal{O}(U_k^2), \quad (4)$$

where we introduced the abbreviation $v_d = 2^{-(d+1)} \pi^{-d/2} \Gamma^{-1}(d/2)$, which is related to the volume of d spheres. It is convenient to remove the explicit Z_k and k dependence by using

¹Note that our convention follows [7] and thus is opposite to the one used by Halpern and Huang in [1, 2] and also by the author of [14]: $t = -t_{\text{HH}}$.

dimensionless scaling variables:

$$\varphi = Z_k^{1/2} k^{1-d/2} \phi, \quad \tilde{\rho} = \frac{1}{2} \varphi^2 = Z_k k^{2-d} \rho, \quad u_k(\varphi) = k^{-d} U_k(\phi). \quad (5)$$

In the same spirit, we write for the cutoff function

$$R_k(q^2) = Z_k k^2 C(q^2/k^2), \quad (6)$$

where $C(w)$ is a dimensionless function of a dimensionless argument, satisfying $C(w \rightarrow 0) \rightarrow 1$ and $C(w \rightarrow \infty) \rightarrow 0$. Rewriting Eq. (4) in terms of these variables and taking the RG time derivative ∂_t on the left-hand side at fixed $\tilde{\rho}$, we obtain the differential equation

$$\partial_t u_k(\tilde{\rho}) = -du_k(\tilde{\rho}) + (d-2+\eta)\tilde{\rho}\dot{u}_k(\tilde{\rho}) - \frac{1}{2}\kappa(2\tilde{\rho}\ddot{u}_k(\tilde{\rho}) + N\dot{u}_k(\tilde{\rho})), \quad (7)$$

where the dot denotes a derivative with respect to the argument $\tilde{\rho}$, and the complete cutoff dependence is contained in

$$\kappa = \kappa(d, \eta; C) = 2v_d \int_0^\infty dw \left[(d-2) \frac{w^{d/2-2} C(w)}{w + C(w)} - \eta \frac{w^{d/2-1} C(w)}{(w + C(w))^2} \right]. \quad (8)$$

We are looking for eigenpotentials, i.e., tangential directions to the RG flow of the scaling form $u_k \sim e^{-\lambda t}$, where λ classifies the possible directions and distinguishes between irrelevant ($\lambda < 0$), marginal ($\lambda = 0$) and relevant ($\lambda > 0$) perturbations away from the Gaussian fixed point. Solutions of this form can be given in terms of the Kummer function M [16]

$$u_k(\tilde{\rho}) = -e^{-\lambda t} \frac{2\kappa r}{d-\lambda} \left[M\left(\frac{\lambda-d}{d-2+\eta}, \frac{N}{2}; \frac{d-2+\eta}{\kappa} \tilde{\rho}\right) - 1 \right]. \quad (9)$$

For given dimension, N , cutoff specification and anomalous dimension, the Halpern-Huang potential (9) depends on two dimensionless parameters: λ and r . The latter sets a “distance” scale along the RG trajectories; since it is an overall factor, the position of possible extrema of u_k are independent of r .

To make contact with the literature, we note that we rediscover the results of Periwal [14] in the limit $\eta = 0$, where the Halpern-Huang result was generalized to arbitrary cutoffs within the Polchinski RG approach [15]. The results of Halpern and Huang are recovered by employing a sharp cutoff, for which κ is related to the volume of the $d-1$ dimensional sphere²:

$$\kappa(d, \eta = 0, C_{sc}) = \frac{\text{vol.}(S^{d-1})}{(2\pi)^d}. \quad (10)$$

²The sharp cutoff limit of Eq. (8) has to be defined carefully; details can be found in [13, 9].

Various representations for the Kummer function $M(\alpha, \beta; x)$ exist in the literature [16]; for further discussion, it is useful to replace the parameter λ by the combination

$$a := 1 + \frac{\lambda - d}{d - 2 + \eta}. \quad (11)$$

Then, Eq. (9) reduces to standard polynomial potentials of degree n in $\tilde{\rho}$ ($2n$ in φ) if $a = -n + 1$; for all such polynomial potentials, the Gaussian fixed point is IR stable. For $a = 1$, the potential vanishes, and for any other value of a , the potential is nonpolynomial. For these cases, the asymptotic behavior for large third argument x is given by an exponential increase

$$M(\alpha, \beta; x) \simeq \frac{\Gamma(\beta)}{\Gamma(\alpha)} x^{(\alpha-\beta)} e^x (1 + \mathcal{O}(x^{-1})). \quad (12)$$

The Gaussian fixed point is UV stable ($\lambda > 0$) for

$$a > -\frac{2 - \eta}{d - 2 + \eta}, \quad (13)$$

(as long as $d - 2 + \eta > 0$). A particularly interesting case is given by the parameter set

$$-1 < a < 0, \quad r < 0, \quad (14)$$

for which the eigenpotential Eq. (9) is nonpolynomial and develops a minimum, inducing spontaneous symmetry breaking.

To conclude our derivation of the Halpern-Huang results, we mention that in the particular case of $N = 1$ there exist (physically admissible) solutions to Eq. (7) which are odd under $\varphi \rightarrow -\varphi$ [5]. The linearized flow equation has also been studied from a different perspective employing its similarity to a Fokker-Planck form [17].

According to the scenario outlined in the introduction, we shall now consider the potentials found in Eq. (9) taken at $t = 0$ ($k = \Lambda$) as the boundary condition for the complete flow equation (3). Provided that the anomalous dimension η is only weakly dependent on k and bounded (as is the case, e.g., for polynomial interactions in $d > 2$), some features can immediately be read off from Eq. (3): for nonpolynomial potentials with exponential asymptotics given by Eq. (12), the denominators on the right-hand side of the flow equation (3) vanish exponentially for large values of ρ . Therefore, $\partial_t U_k(\rho) \rightarrow 0$ for large ρ , and the flow halts, leaving U_k essentially unchanged.

In particular, for symmetry-preserving potentials with a minimum at $\rho = 0$ and $a > 1$, we may expect a rather unspectacular flow: for large ρ , the above argument holds, whereas for small ρ , we may always find a small region where the linearization of the flow equation is a good approximation; there, the Halpern-Huang potential will still be an appropriate approximation. Therefore, these potentials are expected to behave stiffly under the flow.

For potentials with a minimum at nonvanishing ρ (spontaneous symmetry breaking) with $-1 < a < 0$, the asymptotics for large ρ will also stop the flow. However, the flow of

U_k near the nontrivial minimum can be more complicated, since U'_k and U''_k are no longer monotonic functions in this region. To the right of the minimum, these potentials may also be stiff under the flow, but the region around the origin and the minimum appear as a loose end.

These heuristic arguments will be worked out and confirmed in the following section in the large- N limit.

3 RG flow of Halpern-Huang theories in the large- N limit

For solving flow equations for the effective average potential of the type of Eq. (3), several techniques have been developed. Of course, it is always possible to search numerically for solutions by putting the differential equation on a computer; in fact, if one is looking for accurate results, this is the most appropriate option. However, since the potentials under consideration exhibit an exponential increase, straightforward numerics may come to its limits and a clever variable substitution has to be guessed.

Another possibility is to expand the potential in terms of a complete set of functions and decompose the flow equation into differential equations for the k -dependent coefficients (generalized couplings). Here, a choice for a useful set of functions again has to be guessed; obviously, the polynomials as the standard choice are of no use, because the important information is contained in the nonpolynomial nature of the potential.

Therefore, we decide to work in the large- N limit which puts no a priori restrictions on the form of the potential and allows for a complete integration of the flow equation. Of course, the validity of the results for finite values of N can hardly be controlled at this early stage.

3.1 RG flow equation in the large- N limit

In the large- N limit, the RG flow equation (3) for the potential simplifies considerably; here we shall follow the presentation given in [12] and [13]. Not only does the anomalous dimension η vanish [18], but so does the influence of higher derivative terms (Y_k, \dots). Moreover, the Goldstone modes dominate the right-hand side of Eq. (3) and any contribution from the radial mode can be neglected (this essentially changes the order of the differential equation).

For technical reasons, one finally chooses a sharp cutoff function R_k and decides to consider the flow equation for the *derivative* of the potential. In dimensionless variables, the large- N limit of the flow equation reads

$$2\dot{u}_k = -\frac{\partial \dot{u}_k}{\partial t} + \left((d-2)\tilde{\rho} - \frac{2v_d N}{1 + \dot{u}_k} \right) \frac{\partial \dot{u}_k}{\partial \tilde{\rho}}. \quad (15)$$

Of course, this equation can be obtained directly from the sharp-cutoff formulation of the

RG and has already been studied by Wegner and Houghton [4]; further investigations of the Wegner-Houghton approach have been made in [19].

Following [12], this partial differential equation of first order can be solved using the standard method of characteristics and we find that the solution $\dot{u}_k(\tilde{\rho})$ has to satisfy the equation

$$\tilde{\rho} = s(\dot{u}_k) e^{-(d-2)t} - v_d N I(d, t; \dot{u}_k), \quad (16)$$

where $I(d, t; \dot{u}_k)$ is defined by the integral

$$I(d, t; \dot{u}_k) := e^{-(d-2)t} \int_0^{\exp(-2t)} dw \frac{w^{-d/2}}{1 + e^{2t} \dot{u}_k w}. \quad (17)$$

This function is studied in App. A and explicit representations for $d = 3$ and $d = 4$ are given. The function $s(\dot{u}_k)$ is implicitly defined by the equation

$$\dot{u}_\Lambda(s) = \dot{u}_k e^{2t} \implies s \equiv s(\dot{u}_k), \quad (18)$$

where $\dot{u}_\Lambda(s)$ represents the boundary condition for the flow equation at $k = \Lambda$ ($t = 0$); here, s as a variable parametrizes the boundary condition and corresponds to the $\tilde{\rho}$ axis at $t = 0$ in the $\tilde{\rho}, t$ plane. It is exactly Eq. (18) that is to be inserted into Eq. (16), where the nonpolynomial potentials enter the investigation.

Now the route to an explicit solution is clear: (i) we specify the boundary condition via Eq. (18), (ii) insert this and an explicit representation for $I(d, t; \dot{u}_k)$ into Eq. (16), and “solve” (or invert) the resulting equation for $\dot{u}_k(\tilde{\rho})$. However, in practise, some complications are encountered: e.g., inverse functions of such complicated objects as the Kummer function M are not easily obtainable. But the large- N limit comes to the rescue once more as demonstrated in the next subsection.

Let us finally extract the flow of a possible minimum of the potential which is defined by $\dot{u}_k(\tilde{\rho}_{\min}) = 0$; from Eq. (16), we can easily extract that

$$\tilde{\rho}_{\min}(k) = \tilde{\rho}_{\min}(\Lambda) e^{-(d-2)t} - v_d N I(d, t; 0), \quad (19)$$

where $\tilde{\rho}_{\min}(\Lambda)$ denotes the minimum of the potential at $k = \Lambda$ ($t=0$), i.e., the minimum of the Halpern-Huang potential (in the large- N limit); by construction, it is identical to $\tilde{\rho}_{\min}(\Lambda) = s(\dot{u}_k(\tilde{\rho}_{\min}) = 0)$. The function $I(d, t; 0)$ can be read off from Eqs. (A.3) and (A.4) of the appendix ($I(d, t; 0) \equiv i_0(d, t)$). Reinstating dimensionful quantities (cf. Eq. (5)), we find for the flow of a minimum of the potential

$$\rho_{\min}(k) = \rho_{\min}(\Lambda) - \tilde{\rho}_{\text{cr}} (\Lambda^{d-2} - k^{d-2}), \quad \tilde{\rho}_{\text{cr}} := \frac{2v_d N}{d-2}. \quad (20)$$

Here, we introduced a “critical” (dimensionless) field strength $\tilde{\rho}_{\text{cr}}$. Of course, Eq. (20) is well known in the literature [13] and makes no particular reference to the type of potential

under consideration. The only place where the potential type enters is the position of the initial minimum $\rho_{\min}(\Lambda)$. If $\rho_{\min}(\Lambda) > \tilde{\rho}_{\text{cr}}\Lambda^{d-2}$, then the classical as well as the quantum theory exhibit spontaneous symmetry breaking, since $\rho_{\min}(k \rightarrow 0) > 0$; if $\rho_{\min}(\Lambda) < \tilde{\rho}_{\text{cr}}\Lambda^{d-2}$ the quantum theory will preserve $O(N)$ symmetry. Finally, if $\rho_{\min}(\Lambda) = \tilde{\rho}_{\text{cr}}\Lambda^{d-2}$ the classical potential U_Λ is “fine-tuned” in such a way that the theory shows symmetry breaking for finite values of k , but restores $O(N)$ symmetry in the limit $k \rightarrow 0$; additionally, the potential has a vanishing mass term: $M^2 := U'_{k \rightarrow 0}(0) = 0$ (by construction)³.

In standard ϕ^4 theory, the position of the minimum $\rho_{\min}(\Lambda)$ of U_Λ can be chosen at will by an appropriate tuning of the negative mass term and the coupling. By contrast, for Halpern-Huang potentials, once the precise type of the potential is chosen by fixing λ (or a), there is no parameter left for any fine-tuning, since a possible minimum (for theories with $-1 < a < 0$) is independent of the last free parameter r in Eq. (9). The question as to whether a symmetry-breaking quantum theory of the Halpern-Huang potentials exists has to be answered by determining the position of the initial minimum $\rho_{\min}(\Lambda)$. This will also be investigated in the next subsection in the large- N limit.

3.2 Large- N limit of the Halpern-Huang potentials

In our scenario, the Halpern-Huang potential enters the flow equation as its boundary condition at the cutoff. Since the flow equation is considered in the large- N limit, it is not only useful to insert the large- N limit of the Halpern-Huang potential into Eq. (16), but it is mandatory for reasons of consistency. Otherwise, nonleading large- N information would be mixed with large- N behavior, introducing some arbitrariness into this approximation.

From Eq. (9), we read off that the parameter N occurs only in the second argument of the Kummer function. Unfortunately, we could not find any asymptotic expression for the Kummer function with large second argument in the literature. Instead of investigating this limit in terms of some appropriate series or integral representation, which might involve awkward interchanges of limiting processes, we shall use a more physically motivated approach: since the Kummer function was identified with the tangential direction to the RG flow in the vicinity of the Gaussian fixed point, its large- N approximation should naturally be deducible from a large- N study of the same subject. In other words, the desired function has to be a solution to the linearized flow equation in the large- N limit.

For technical reasons, we again turn to the derivative of the potential with respect to $\tilde{\rho}$. Then, the desired differential equation is obtained by linearizing Eq. (15). Looking for potentials which satisfy the eigenpotential scaling condition $-\partial_t \dot{u}_k = \lambda \dot{u}_k$, the large- N Halpern-Huang equation reads

$$\frac{d\dot{u}_\Lambda(\tilde{\rho})}{d\tilde{\rho}} = -\frac{a}{\tilde{\rho} - \tilde{\rho}_{\text{cr}}} \dot{u}_\Lambda(\tilde{\rho}), \quad (21)$$

where we again traded λ for the parameter a as defined in Eq. (11). Additionally, we made use of the “critical” field strength $\tilde{\rho}_{\text{cr}}$ defined in Eq. (20).

³In the language of statistical mechanics, the theory is exactly at the critical temperature.

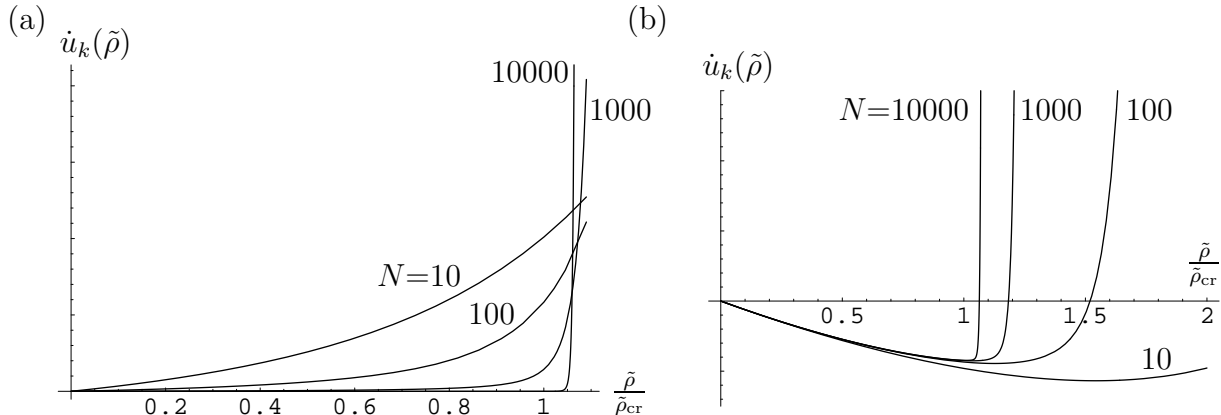


Figure 1: Large- N behavior of (a) a symmetry-preserving ($a = 2$) and (b) a symmetry-breaking Halpern-Huang potential ($a = -1/2$) in $d = 4$ dimensions versus $\tilde{\rho}/\tilde{\rho}_{cr}$. For $N \rightarrow \infty$, the potentials develop a “wall” at $\tilde{\rho} = \tilde{\rho}_{cr}$. In (a), the normalization of the potentials is appropriately adapted for illustrative purposes.

Equation (21) can easily be solved for the various boundary conditions; let us begin with a symmetry-preserving potential satisfying $\dot{u}_\Lambda(\tilde{\rho}) > 0$ and $a > 0$:

$$\dot{u}_\Lambda(\tilde{\rho}) = v_+ \left(\frac{\tilde{\rho}_{cr}}{\tilde{\rho}_{cr} - \tilde{\rho}} \right)^a, \quad v_+ := \dot{u}_\Lambda(0), \quad (22)$$

where the initial value v_+ also satisfies $v_+ > 0$. The parameter v_+ is, of course, the large- N analogue of the distance parameter r in Eq. (9), $r \rightarrow (N/4)v_+$.

At first sight, one may doubt the validity of Eq. (22) as the large- N limit of Eq. (9), because it diverges at the critical field strength $\tilde{\rho} \rightarrow \tilde{\rho}_{cr}$. Nevertheless, this indeed reflects the behavior of the Kummer function for large second argument, as can be checked numerically (see Fig. 1(a)) [20]. The critical field strength $\tilde{\rho}_{cr}$ marks the point where the asymptotic exponential increase (cf. Eq. (12)) sets in; and for larger values of N , the slope increases without bound⁴. Of course, a potential that diverges for finite values of its argument is usually considered as inadmissible in field theory (see, e.g., [4] and [21]); however, in the present case, we take the viewpoint that this potential wall at $\tilde{\rho} = \tilde{\rho}_{cr}$ only symbolizes the exponential increase of the potential for finite values of N .

Let us now turn to the solution of Eq. (21) for the symmetry-breaking potentials with ($-1 < a < 0$). In the inner region of the potential where $\dot{u}_\Lambda < 0$, we obtain the solution

$$\dot{u}_\Lambda(\tilde{\rho}) = v_- \left(\frac{\tilde{\rho}_{cr} - \tilde{\rho}}{\tilde{\rho}_{cr}} \right)^{-a}, \quad v_- := \dot{u}_\Lambda(0), \quad |a| = -a, \quad (23)$$

where the initial value this time satisfies $v_- < 0$. The derivative of the large- N Halpern-Huang potential has a zero at $\tilde{\rho} = \tilde{\rho}_{cr}$, so that the potential itself exhibits a minimum at

⁴This might be the reason why there is no large- N limit of the Kummer function in the literature: it cannot be defined for arbitrary values of $\tilde{\rho}$.

this position. Now let us turn to the large- N limit of the symmetry-breaking potential to the right of the minimum $\tilde{\rho} > \tilde{\rho}_{\text{cr}}$ where $\dot{u}_\Lambda > 0$. As a matter of fact, the unique solution of Eq. (21) increases only very slowly, $\dot{u}_\Lambda \sim \tilde{\rho}^{|a|}$ for $\tilde{\rho} \rightarrow \infty$ and $-1 < a < 0$. This does certainly not reflect the expected exponential increase; hence this solution has to be discarded. Even if a more appropriate solution existed for $\tilde{\rho} > \tilde{\rho}_{\text{cr}}$, we would not be able to match them properly at $\tilde{\rho} = \tilde{\rho}_{\text{cr}}$, because the second derivative of the potential diverges at this point. In view of the results for the symmetry-preserving potential and owing to the fact that the asymptotic exponential increase for both types of potentials is the same (see Eq. (12)), the only possibility for the large- N limit is to continue the potential at $\tilde{\rho} = \tilde{\rho}_{\text{cr}}$ by a potential wall at $\tilde{\rho} = \tilde{\rho}_{\text{cr}} + 0^+$. Again, a numerical analysis of the full Kummer function for large N confirms this conjecture, as is depicted in Fig. 1(b).

This concludes our large- N analysis of the Halpern-Huang potential; note that both types of the potential are formally equivalent, so that we combine them in the notation

$$\dot{u}_\Lambda(\tilde{\rho}) = v \left(\frac{\tilde{\rho}_{\text{cr}}}{\tilde{\rho}_{\text{cr}} - \tilde{\rho}} \right)^a, \quad v := \dot{u}_\Lambda(0), \quad (24)$$

where v stands for v_+ or v_- and $a \in \mathbb{R}$. Actually, this also covers a ϕ^4 potential for $a = -1$.

3.3 Nonperturbative evolution of the Halpern-Huang potentials

Now we are in a position to study the flow of the Halpern-Huang potentials from $k = \Lambda$ into the infrared regime $k \rightarrow 0$ in the large- N limit. The missing piece of information to be inserted into Eq. (16) is given by the inverse of Eq. (24):

$$\dot{u}_\Lambda(s) = v \left(\frac{\tilde{\rho}_{\text{cr}}}{\tilde{\rho}_{\text{cr}} - s} \right)^a, \quad \stackrel{(18)}{\Rightarrow} \quad s(\dot{u}_k) = \tilde{\rho}_{\text{cr}} \left[1 - \left(\frac{v}{\dot{u}_k e^{2t}} \right)^{1/a} \right]. \quad (25)$$

Employing the representation (A.3) for the function $I(d, t; \dot{u}_k)$, we find that the derivative of the potential has to satisfy the equation

$$0 = (\tilde{\rho}_{\text{cr}} - \tilde{\rho}) - \tilde{\rho}_{\text{cr}} \left(\frac{v}{\dot{u}_k(\tilde{\rho})} \right)^{1/a} e^{-(\lambda/a)t} + \frac{d-2}{2} \tilde{\rho}_{\text{cr}} \dot{u}_k(\tilde{\rho}) J(d, t; \dot{u}_k), \quad (26)$$

where the function $J(d, t; \dot{u}_k)$ is defined in Eq. (A.5). Finally, Eq. (26) has to be solved for $\dot{u}_k(\tilde{\rho})$, which we shall do in the limit $k \rightarrow 0$ ($t \rightarrow -\infty$) for various cases in order to obtain the complete quantum effective potential. The following consideration will serve as a guide to the necessary approximations: at the cutoff $k = \Lambda$, the dimensionful potential is of the order of the cutoff $U'_\Lambda \sim \Lambda^2$. For small deviations from the cutoff, $k/\Lambda \lesssim 1$, the potential scales according to the linearized flow equation (Halpern-Huang equation): $\dot{u}_k \sim e^{-\lambda t}$. Then, the dimensionful potential scales as $U'_k \sim e^{-(\lambda-2)t} \sim e^{-(d-2)at}$. Therefore, if $a > 0$ (symmetry-preserving potentials), U'_k increases as we approach the infrared, whereas if $a < 0$ (symmetry-breaking potentials), U'_k decreases towards the infrared. Of course, this argument holds strictly close to $k \simeq \Lambda$ only, but it turns out to reproduce the unique consistent approximation schemes for extracting analytical results.

3.3.1 Symmetry-preserving potentials in $d = 4$

Let us first consider the $d = 4$ potentials with $a > 0$ and $U'_\Lambda > 0$ that exhibit no symmetry breaking at the cutoff. Employing Eq. (A.6) and reinstating dimensionful quantities via Eq. (5), Eq. (26) reads, after neglecting terms of order k^2/U'_k in the limit $k \rightarrow 0$ ($U'_{k \rightarrow 0} \equiv U'$):

$$0 = \tilde{\rho}_{\text{cr}} U'(\rho) \ln \left(1 + \frac{\Lambda^2}{U'(\rho)} \right) - \rho - \tilde{\rho}_{\text{cr}} \Lambda^2 \left(\frac{U'_\Lambda(0)}{U'(\rho)} \right)^{1/a}, \quad (27)$$

where $U'_\Lambda(0) = v_+ \Lambda^2 \equiv M_\Lambda^2$ denotes the mass of the theory at the cutoff. Let us study Eq. (27) in two limits: first, at ρ close to $\tilde{\rho}_{\text{cr}} \Lambda^2$, and secondly at the origin $\rho \rightarrow 0$.

At ρ close to the potential wall at $\tilde{\rho}_{\text{cr}} \Lambda^2$, the potential diverges, and we can approximate $\Lambda^2 \ll U'(\rho \rightarrow \tilde{\rho}_{\text{cr}} \Lambda^2)$, leading us to

$$U'(\rho \rightarrow \tilde{\rho}_{\text{cr}} \Lambda^2) = U'_\Lambda(0) \left(\frac{\tilde{\rho}_{\text{cr}} \Lambda^2}{\tilde{\rho}_{\text{cr}} \Lambda^2 - \rho} \right)^a. \quad (28)$$

In this limit, the effective potential $U \equiv U_{k \rightarrow 0}$ remains formally identical to the large- N Halpern-Huang potential (cf. Eq. (22))! This confirms our heuristic argument that the potential behaves stiffly under the flow in the region where it increases exponentially.

Concerning the opposite limit $\rho \rightarrow 0$, there would be no mass renormalization at all, if Eq. (28) were also correct in this limit, $M^2 := U'(0) \stackrel{?}{=} U'_\Lambda(0) = M_\Lambda^2$. However, in this limit, the approximation $\Lambda^2 \ll U'$ no longer holds, and instead we deduce from Eq. (27) the transcendental equation

$$1 = v_+ \left(\frac{M^2}{M_\Lambda^2} \right)^{(a+1)/a} \ln \left(1 + \frac{1}{v_+} \frac{M_\Lambda^2}{M^2} \right). \quad (29)$$

Therefore, the mass renormalization is governed by the only free parameter of the theory, v_+ : for large v_+ , there is effectively no renormalization, whereas the renormalized mass M^2 exceeds the “classical” mass M_Λ^2 for $v_+ \lesssim 1$. Typical values are $M^2 \simeq 10 M_\Lambda^2$ for $v_+ = 0.01$ and $a = 2$; for larger values of the RG trajectory parameter a , the mass shift even increases: $M^2 \simeq 100 M_\Lambda^2$ for $v_+ = 0.01$ and $a = 20$. The M^2/M_Λ^2 relation is plotted against v_+ for various a in Fig. 2(a).

By reintroducing the cutoff again via $M_\Lambda^2 = v_+ \Lambda^2$, Eq. (29) can be interpreted differently by writing

$$v_+ = \left(\frac{M^2}{\Lambda^2} \right)^{a+1} \left[\ln \left(1 + \frac{\Lambda^2}{M^2} \right) \right]^a. \quad (30)$$

This equation tells us that the physical mass of the theory in the infrared can easily be much smaller than the cutoff by tenth of orders of magnitude, provided that v_+ is correspondingly small. Since v_+ sets the distance scale on the RG trajectory, the demand for a small value of v_+ is consistent with our scenario: if we leave the Gaussian fixed point with a very tiny

perturbation $\sim v_+$ at the high-energy scale Λ , it is only *natural* to arrive at a low-energy theory with a similarly tiny mass compared to the cutoff. Moreover, consistency of our scenario requires v_+ to be small in order to justify the linearization of the flow equation in deriving the Halpern-Huang result.

To summarize, the symmetry-preserving Halpern-Huang potential qualitative does not change its form during the flow into the infrared; in particular, no symmetry breaking occurs. Only the slope of the potential at the origin of the theory increases for $k \rightarrow 0$, which corresponds to a mass renormalization.

3.3.2 Symmetry-breaking potentials in $d = 4$

Let us begin with a dimension-independent statement referring to the position of the minimum of symmetry-breaking Halpern-Huang potentials with $-1 < a < 0$: in Subsec. 3.2 we learned that the position of the minimum of the Halpern-Huang potentials in the large- N limit is independent of the parameters a and v_- : $\tilde{\rho}_{\min}(\Lambda) = \tilde{\rho}_{\text{cr}}$, or in dimensionful quantities: $\rho_{\min}(\Lambda) = \tilde{\rho}_{\text{cr}}\Lambda^{d-2}$. According to the discussion following Eq. (20), the Halpern-Huang potentials are “fine-tuned” in the sense that the minimum vanishes exactly in the infrared limit $k \rightarrow 0$:

$$\rho_{\min}(k) = \tilde{\rho}_{\text{cr}}k^{d-2}. \quad (31)$$

Therefore, there is no symmetry breaking in the full quantum theory of Halpern-Huang potentials in the large- N limit. Moreover, since $M^2 = U'(\rho = 0) \equiv U'_{k=0}(\rho = 0) = 0$, the potential is flat at the origin and the renormalized quantum theory is massless.

Following the line of argument given below Eq. (26), the inner region of the potential where $U'_k < 0$ decreases towards the infrared; hence we approximate $|U'_k|/\Lambda^2 \ll 1$ in Eq. (26) and obtain the transcendental equation in $d = 4$:

$$-U'_k = k^2 - \Lambda^2 \exp\left(-\frac{\tilde{\rho}_{\text{cr}}k^2 - \rho}{\tilde{\rho}_{\text{cr}}(-U'_k)}\right). \quad (32)$$

Here we can read off that $|U'_k|$ is always smaller than k^2 . This reflects the approach to convexity of the inner part of the effective potential.

To summarize, we have found, on the one hand, that the originally nontrivial minimum of the potential moves to the origin during the flow; the inner region of the potential shrinks to a point. On the other hand, we know from the preceding subsection that the potential wall at $\rho = \tilde{\rho}_{\text{cr}}\Lambda^2 + 0^+$ does not change its position under the flow. It remains to be investigated what happens in between the minimum and the potential wall. Unfortunately, we cannot answer this question by the large- N version of the flow equation, because we do not have a boundary condition for this region. At the cutoff $k = \Lambda$, the inner region borders directly at the potential wall; hence, there is no “in-between” that could serve as a boundary condition. Of course, it is plausible to assume that the potential may interpolate smoothly between the origin with zero slope and the potential wall at $\rho = \tilde{\rho}_{\text{cr}}\Lambda^2 + 0^+$ with infinite slope. But alternatively, the potential can also remain flat for $\rho \in [0, \tilde{\rho}_{\text{cr}}\Lambda^2]$, resembling a particle-in-a-box potential.

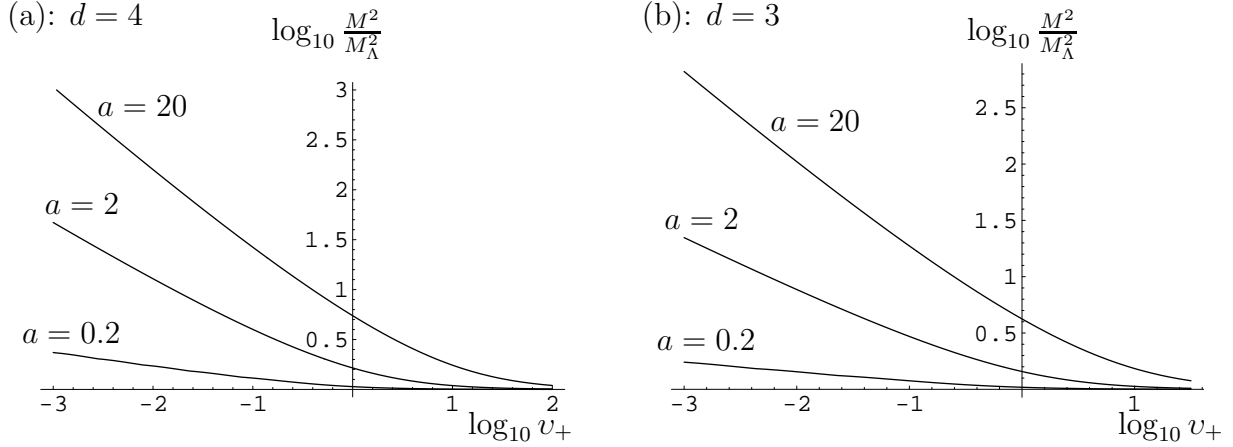


Figure 2: Double-logarithmic plot of the renormalized-to-cutoff mass ratio depending on the RG distance parameter v_+ for (a) $d = 4$ (cf. Eq. (29)) and (b) $d = 3$ (cf. Eq. (33)) for various values of a .

Our ignorance about that part of the potential is unfortunately accompanied by our inability to predict the mass of the radial mode; but this should not come as a surprise, since the large- N limit neglects the radial mode anyway.

3.3.3 Effective potentials in $d = 3$

The investigation of the various types of potentials in $d = 3$ proceeds analogously to the $d = 4$ case with almost identical results. In particular, the symmetry-breaking potentials offer no new information: the inner region shrinks to a point, while the potential minimum moves to zero for $k \rightarrow 0$, and the potential wall remains at $\rho = \tilde{\rho}_{\text{cr}}\Lambda$. In between, no confirmed statement can be made within the large- N limit, since no boundary condition governs this part of the potential.

For symmetry-preserving potentials with $a > 0$, the potential again remains in the same form as at the cutoff for values of ρ close to the potential wall at $\tilde{\rho}_{\text{cr}}\Lambda$ (cf. Eq. (28) with Λ^2 replaced by Λ).

Close to the origin $\rho \rightarrow 0$, the shape of the potential is modified; this is reflected by a mass renormalization. Employing the same line of argument as given above in $d = 4$, and using Eq. (A.7), we find the $d = 3$ analogue of Eq. (29):

$$1 = \sqrt{v_+} \left(\frac{M}{M_\Lambda} \right)^{(a+2)/a} \arctan \frac{1}{\sqrt{v_+}} \frac{M_\Lambda}{M}. \quad (33)$$

Again, we find that there is no mass renormalization for large values of v_+ ; corrections for small values of v_+ are plotted in Fig 2(b).

4 Conclusions

In the present paper, we have investigated the RG flow of particular nonpolynomial potentials for $O(N)$ symmetric scalar theories using the effective-average-action method. These Halpern-Huang potentials arise from small relevant perturbations at the Gaussian fixed point as tangential directions to the RG flow. Apart from serious, unresolved problems with the continuum limit of these potentials, we were able to follow the flow from a given ultraviolet scale Λ down to the nonperturbative infrared; for this, a number of approximations have been made which are only under limited control. In a first step, we have neglected the influence of possible derivative couplings on the flow of the potential.

Secondly, assuming that the anomalous dimension is only weakly dependent on k and bounded, the qualitative features of the flow could already be guessed from the form of the flow equation: this is because the exponential increase of the potentials essentially causes the flow to stop for large enough field values. Therefore, the form of the potentials was recognized as stiff under the flow; only the loose ends of the potential near the origin or possible extrema make room for more diversified behavior.

These considerations have been verified explicitly in the large- N limit of the system. In this limit, the exponential increase of the potentials is represented by a potential wall. The potential close to the wall and the wall itself remain unchanged even in the far infrared. Those potentials with an $O(N)$ symmetric ground state ($a > 0$) at the cutoff preserve this symmetry down to $k \rightarrow 0$. Our main result for such potentials is summarized in Eqs. (29), (30) and (33), where the particular form of the mass renormalization is stated. Contrary to polynomial scalar interactions where the mass varies $\sim k^2$ during the flow, the Halpern-Huang potentials exhibit corrections which are governed by the RG distance parameter v_+ . In particular, if one demands that a renormalized (infrared) mass differ by several orders of magnitude from the cutoff scale Λ , the bare parameters of a *polynomial* theory at the cutoff scale have to be fine-tuned accurately to several decimal places. By contrast, to achieve such a separation of mass scales with a *nonpolynomial* Halpern-Huang potential, an adjustment of the RG distance parameter at the cutoff to some small value is required with much less precision. Additionally, the smallness of this value arises naturally, if the (unknown) perturbation at the Gaussian fixed point is tiny. The (symmetry-preserving) Halpern-Huang potentials thus has no problem of *naturalness*. Owing to the general properties of the complete flow equation mentioned above, we believe that these properties of the symmetry-preserving potentials in the large- N limit also hold for finite values of N .

The status of the large- N limit is certainly different for Halpern-Huang potentials which offer spontaneous symmetry breaking ($-1 < a < 0$). These potentials exhibit the remarkable property that the nontrivial minimum persists for any finite value of k but vanishes in the complete quantum theory for $k \rightarrow 0$ in the large- N limit; the $O(N)$ symmetry is restored and the potential becomes flat near the origin. The coincidence between the position of the minimum and the critical value of the field strength may finally be ascribed to the formal resemblance between the large- N flow equation and its linearized version determining the Halpern-Huang potentials. Since the complete flow equation is much more complex, it appears rather improbable that this property continues to hold for

finite N . Therefore, whether or not spontaneous symmetry breaking occurs in the quantum version of the Halpern-Huang potential at finite N remains an open question. The present investigation at least observes a tendency of the system to restore $O(N)$ symmetry. This is in concordance with [22], where a one-loop calculation for the effective potential reveals a restoration of $O(N)$ symmetry for potentials with $(-1 < a \lesssim -0.585)$.

In this context, a possible application of the Halpern-Huang potentials to the Higgs sector of the standard model is still questionable. Even if a quantum version of the potential with spontaneous symmetry breaking exists, the naturalness of the scalar sector alone is not sufficient to solve the hierarchy problem. This is because the (standard) Yukawa coupling to the fermions leads to large scalar mass renormalizations by fermion loops. Therefore, some appropriate nonpolynomial interaction has to be chosen, also in this sector. Nevertheless, the price to be paid would not be too high, because not only the hierarchy problem could be circumvented without additional degrees of freedom, but also the problem of “triviality” would be evaded.

From an intuitive point of view, the fact that the form of the potential is stable under the RG flow appears to be disappointing: since the potential remains inherently nonpolynomial, it is impossible to make contact with a would-be classical behavior that is determined by only a few (polynomial) terms. The latter is usually expected at large distances. For example, merely for very weak fields do the first terms in a Taylor expansion of the Kummer function represent a good approximation. For stronger fields, the application of the Halpern-Huang potentials might therefore be limited in this sense.

From a technical viewpoint, our calculations hold for $d > 2$. We have given explicit results for $d = 3$ and $d = 4$, and generalizations to higher dimensions are straightforward. The limiting case $d = 2$ has to be treated with great care for several reasons. First of all, finite N results may only be trusted if the flow of the anomalous dimension η is taken into account; at least in the case of polynomial potentials, this turned out to be obligatory [13] in order to obtain a good picture of the Kosterlitz-Thouless transition. Furthermore, the limit $d \rightarrow 2$ of the Halpern-Huang potentials offers several possibilities. It has already been observed variously in the literature (see, e.g., [17]), that the Sine-Gordon as well as the Liouville potentials solve the linearized flow equation in $d = 2$. In fact, as can be easily shown with the aid of some identities of [16], both types of potentials arise as limiting cases of the Halpern-Huang potentials for $N = 1$ in combination with the $\phi \rightarrow -\phi$ odd solution of the linearized flow equation: to be precise, the Sine-Gordon potential is recovered in the limit $d \rightarrow 2^+$ for $\lambda > 2$, whereas the Liouville potential is obtained by taking the limit $d \rightarrow 2^-$ for $0 < \lambda < 2$.

As far as the Liouville theory is concerned, further similarities to the present results for the symmetry-preserving potentials are visible. In [23], the Liouville potential has also been found to behave stiffly under the RG flow for similar reasons as in the present case. In particular, quantum Liouville theory appears to equal classical Liouville theory, except for a flow of the central charge by one unit and a modified mass parameter. These similarities confirm the viewpoint that the Halpern-Huang potentials can be regarded as higher-dimensional analogues of Liouville theory.

Acknowledgement

The author wishes to thank W. Dittrich for helpful conversations and for carefully reading the manuscript. Useful discussions with R. Shaisultanov are also gratefully acknowledged.

Appendix

A Integrals for the large- N flow equation

In this appendix, we present some details about the function $I(d, t; \dot{u}_k)$ appearing in the solution (16) to the flow equation (15); this function is defined as

$$I(d, t; \dot{u}_k) := e^{-(d-2)t} \int_0^{\exp(-2t)} dw \frac{w^{-d/2}}{1 + e^{2t} \dot{u}_k w}. \quad (\text{A.1})$$

Substituting $w = \exp[-2(T + t)]$, we arrive at the form

$$I(d, t; \dot{u}_k) = 2 \int_0^{-t} dT \frac{e^{(d-2)T}}{1 + \dot{u}_k e^{-2T}}, \quad (\text{A.2})$$

where $t = \ln k/\Lambda$ is always nonpositive: $t \in]-\infty, 0]$. Separating the zeroth-order term of a Taylor expansion of the integrand, we find the convenient representation

$$I(d, t; \dot{u}_k) = i_0(d, t) - \dot{u}_k J(d, t; \dot{u}_k), \quad (\text{A.3})$$

with the auxiliary functions $i_0(d, t)$ and $J(d, t; \dot{u}_k)$ defined by

$$i_0(d, t) := \frac{2}{d-2} (e^{-(d-2)t} - 1), \quad (\text{A.4})$$

$$J(d, t; \dot{u}_k) := 2 \int_0^{-t} dT \frac{e^{(d-4)T}}{1 + \dot{u}_k e^{-2T}}. \quad (\text{A.5})$$

Note that $i_0, J \geq 0$ for $t \leq 0$ and $\dot{u}_k > -1$. The explicit form of J depends on the spacetime dimension. For $d = 4$, the integral can easily be evaluated by standard means, yielding

$$J(4, t; \dot{u}_k) = \ln \frac{e^{-2t} + \dot{u}_k}{1 + \dot{u}_k}. \quad (\text{A.6})$$

In $d = 3$, we take care of the possibility of a nontrivial minimum (spontaneous symmetry breaking) and find to the right of a possible minimum

$$J(3, t; \dot{u}_k > 0) = -\frac{2}{\sqrt{\dot{u}_k}} \left(\arctan \frac{1}{\sqrt{\dot{u}_k}} - \arctan \frac{e^{-t}}{\sqrt{\dot{u}_k}} \right). \quad (\text{A.7})$$

In the “inner” region to the left of a possible minimum, we obtain

$$J(3, t; \dot{u}_k < 0) = \frac{2}{\sqrt{-\dot{u}_k}} \left(\text{Artanh} \frac{1}{\sqrt{-\dot{u}_k}} - \text{Artanh} \frac{e^{-t}}{\sqrt{-\dot{u}_k}} \right), \quad (\text{A.8})$$

where $\dot{u}_k > -1$ for reasons of consistency.

References

- [1] K. Halpern and K. Huang, Phys. Rev. Lett. **74**, 3526 (1995).
- [2] K. Halpern and K. Huang, Phys. Rev. D **53**, 3252 (1996).
- [3] A.I. Larkin and D.E. Khumel’nitskii, Sov. J. Nucl. Phys. **29**, 1123 (1969); K.G. Wilson, Phys. Rev. Lett. **28**, 248 (1972).
- [4] F.J. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973).
- [5] V. Branchina, [hep-ph/0002013](#) (2000).
- [6] K. Langfeld and H. Reinhardt, Mod. Phys. Lett. A **13**, 2495 (1998); R.F. Langbein, K. Langfeld, H. Reinhardt, L. v. Smekal, Mod. Phys. Lett. A **11**, 631 (1996).
- [7] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
- [8] T.R. Morris, Phys. Rev. Lett. **77**, 1658 (1996); K. Halpern and K. Huang, Phys. Rev. Lett. **77**, 1659 (1996).
- [9] C. Bagnuls and C. Bervillier, [hep-th/0002034](#) (2000).
- [10] N. Tetradis and C. Wetterich, Nucl. Phys. B **422**, 541 (1994).
- [11] P. Hasenfratz, in Proc. *Advanced School of Non-Perturbative Quantum Field Physics*, edited by M. Asorey and A. Dobado, Singapore, World Scientific, 1998, [hep-lat/9803027](#).
- [12] N. Tetradis and D. F. Litim, Nucl. Phys. B **464**, 492 (1996) [[hep-th/9512073](#)].
- [13] J. Berges, N. Tetradis and C. Wetterich, HD-THEP-00-26, [hep-ph/0005122](#) (2000).
- [14] V. Periwai, Mod. Phys. Lett. A **11**, 2915 (1996).
- [15] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
- [16] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, (1964).
- [17] A. Bonanno, Phys. Rev. D **62**, 027701 (2000).
- [18] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Oxford University Press, (1989).
- [19] J. Comellas and A. Travesset, Nucl. Phys. B **498**, 539 (1997).
- [20] The Kummer functions (`Hypergeometric1F1[a,b,z]`) can numerically as well as partly algebraically be treated by Mathematica, Version 4.0.1.0, Wolfram Research, Champaign (1999).
- [21] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B270**, 687 (1986).
- [22] K. Halpern, Phys. Rev. D **57**, 6337 (1998).
- [23] M. Reuter and C. Wetterich, Nucl. Phys. B **506**, 483 (1997).